NASA CR-152,318

NASA-CR-152318
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(NASA-CR-152318) AN INVESTIGATION OF THE TURBULENCE SCALE TENSOR IN A FLAT-PLATE BOUNDARY LAYER Final Report (Aeronautical Research Associates of Princeton) 46 p HC A03/MF A01 CSCL 20D G3/34

N79-33436

Unclas 38830

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N79-33436#

A.R.A.P. REPORT NO. 378

AN INVESTIGATION OF THE TURBULENCE SCALE TENSOR IN A FLAT-PLATE BOUNDARY LAYER

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January 1979

FINAL REPORT

Prepared for
National Aeronautics and Space Administration
Ames Research Center
Moffett Field, California 94035

CONTRACT NO. NAS2-8014

ABSTRACT

A differential equation for the scale tensor in turbulent flow is developed from basic considerations and applied to the flow of a constant-density fluid in the boundary layer on a flat plate. Results from preliminary runs of a computer implementation are discussed.

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1.- INTRODUCTION

The method of analysis of turbulent flows known as full second-order closure modeling has been applied with great success to a variety of flows by A.R.A.P. and others. However, one weak point has remained in this method; that is, the determination of the scale which is an important factor in many of the models. Much of the work has used an empirical function to determine the scale. Although this has worked well in many cases, it requires reevaluation whenever a novel situation is considered.

Another approach has been to use a differential equation to follow the evolution of the scale. This too has been reasonably successful in many cases, but it requires considerably more modeling (and evaluation of modeling parameters) and the results achieved have not been such that universal validity is indicated.

One way of arriving at such a differential equation starts by considering a two-point velocity correlation. This leads naturally to the consideration of a scale tensor, measuring the correlation length for the various pairs of fluctuations of velocity components. It is the contraction (or trace) of this tensor that serves as a measure of the scale in the modeling.

It is evident that much of the information contained in the tensor is lost in the contraction process. During the past year, a computer program to solve the differential equations for the whole scale tensor has been developed at A.R.A.P. for Ames Research Center of NASA under Contract NAS2-8014, as an extension of an effort to develop and understand turbulence modeling, especially as applied to compressible boundary layers. It is hoped that an analysis of the results of this program will lead to a better understanding of the concept of scale in turbulence modeling and of the turbulence itself. In order to keep this first effort in the area at a reasonable level, the project has been restricted to the flow of a constant-density fluid in the boundary layer on a flat plate. This, of course, has the advantage that such flows are well understood experimentally.

Much of the theory upon which the program is based was developed with the support of the Air Force Office of Scientific Research under Contract F44620-76-C-0048.

This paper presents a derivation of the equations that are integrated by the program, with a description of the modeling entailed, and includes a description of the results of some preliminary runs. A companion paper, reference 1, serves as a program manual with descriptions of the numerical methods, operating instructions, and so forth. In an attempt to make each paper self-contained, a certain amount of material appears in both.

General tensor notation is used in much of the analysis. This is done to emphasize that the modeling is intended to apply in any (constant-density) flow situation using any coordinate system. It is valid to think of the equations as Cartesian, with no distinction between subscripts and superscripts, with all derivatives interpreted as simple partial derivatives, and with the metric tensor, g_{ij}, interpreted as the Kronecker delta, δ_{ij}. (Note that $\delta_{\ell\ell} = 3$, by the summation convention.) The subscript t denotes a derivative with respect to time.

The next section describes the modeling used in the equation for the Reynolds stress, and the following section treats the derivation of the scale equation. In section 4, the reduction of these equations to the case of the boundary layer on a flat plate is presented. Finally, section 5 discusses the evaluation of the modeling parameters and the results of the computer calculations made so far.

It is a pleasure to acknowledge the help of Morris Rubesin of NASA Ames Research Center. His encouragement and support over the course of several years have been instrumental in carrying out this project. We also thank our colleague, Ashok Varma, who made significant contributions to several aspects of the analysis.

2.- THE BASIC MODELS

For a fluid of constant density, the equations of motion are

Continuity

$$u^{\ell}_{,\ell} = 0 \tag{1}$$

Momentum

$$u_{i_t} + u^{\ell}u_{i,\ell} = -\frac{p_{,i}}{\rho} + \nu u_{i}{}^{,\ell}{}_{,\ell} \tag{2}$$

In this instance, it is convenient to rewrite equation (2) as

$$u_{i_t} + (u^{\ell}u_i)_{,\ell} = -P_{,i} + \nu u_{i}{}^{,\ell}{}_{,\ell} \tag{3}$$

where equation (1) has been invoked to modify the second term and P has been substituted for p/ρ . P , sometimes called the "kinematic pressure," is loosely called just the "pressure" in what follows. The variables are written as sums of means, denoted by bars, and fluctuations, denoted by primes:

$$u_i = \bar{u}_i + u_i' \tag{4}$$

$$P = \bar{P} + P' \tag{5}$$

The fluctuations then are the deviations from the means, and by definition the mean of a fluctuation is zero. It is assumed that the operation of taking the mean or averaging is interchangeable with differentiation.

The substitution of equations (4) and (5) into equations (1) and (3) gives

$$\bar{u}^{\ell}_{,\ell} + u'^{\ell}_{,\ell} = 0 \tag{6}$$

$$\bar{u}_{i_t} + u'_{i_t} + (\bar{u}^\ell \bar{u}_i)_{,\ell} + (\bar{u}^\ell u'_i)_{,\ell} + (u'^\ell \bar{u}_i)_{,\ell} + (u'^\ell u'_i)_{,\ell} =$$

$$= - \bar{P}_{,i} - P'_{,i} + \nu \bar{u}_i{}^{,\ell}{}_{,\ell} + \nu u'_i{}^{,\ell}{}_{,\ell} \tag{7}$$

Taking the mean of these two equations results in

$$\bar{u}^\ell{}_{,\ell} = 0 \tag{8}$$

$$\bar{u}_{i_t} + (\bar{u}^\ell \bar{u}_i)_{,\ell} + (\overline{u'^\ell u'_i})_{,\ell} = - \bar{P}_{,i} + \nu \bar{u}_i{}^{,\ell}{}_{,\ell} \tag{9}$$

Equation (8) is identical in form to equation (1), but equation (9) has a term with no counterpart in equation (3). This term, involving the correlation of fluctuations, is often called the turbulent stress or Reynolds stress.

For engineering purposes, the main interest lies in determining the mean flow; to do so involves determining the Reynolds stress. Until about ten years ago, the only practical approach was to model the extra term in equation (9) in terms of the mean flow (using Prandtl's "mixing length" concept, or "eddy viscosity"). This technique, which is still very useful in many circumstances, is called first-order closure. However, it is recognized that in some circumstances this is not, indeed, cannot be, sufficient. The next step then is to derive the equation that determines the evolution of the Reynolds stress.

Subtracting equations (8) and (9) from equations (6) and (7) gives

$$u'^\ell{}_{,\ell} = 0 \tag{10}$$

$$u'_{i_t} + (\bar{u}^\ell u'_i)_{,\ell} + (u'^\ell \bar{u}_i)_{,\ell} + (u'^\ell u'_i)_{,\ell} - (\overline{u'^\ell u'_i})_{,\ell} =$$

$$= - P'_{,i} + \nu u'_i{}^{,\ell}{}_{,\ell} \tag{11}$$

Rewrite equation (11) with j substituted for i and multiply by u_i' . Add the result to the original equation (11) multiplied by u_j' . Making use of equations (8) and (10) and averaging, the result is

$$(\overline{u_i'u_j'})_t + \bar{u}^\ell(\overline{u_i'u_j'})_{,\ell} + \overline{u'^\ell u_i'}\ \bar{u}_{j,\ell} + \overline{u'^\ell u_j'}\ \bar{u}_{i,\ell} + (\overline{u'^\ell u_i'u_j'})_{,\ell} =$$

$$= - \overline{u_i'P'_{,j}} - \overline{u_j'P'_{,i}} + \nu\left(\overline{u_i'u_j'{}^{,\ell}{}_{,\ell}} + \overline{u_j'u_i'{}^{,\ell}{}_{,\ell}}\right) \tag{12}$$

Using the rule for the derivative of a product, the right-hand side of this equation can be rearranged to give

$$(\overline{u_i'u_j'})_t + \bar{u}^\ell(\overline{u_i'u_j'})_{,\ell} + \overline{u'^\ell u_i'}\ \bar{u}_{j,\ell} + \overline{u'^\ell u_j'}\ \bar{u}_{i,\ell} + (\overline{u'^\ell u_i'u_j'})_{,\ell} =$$

$$= -(\overline{P'u_i'})_{,j} - (\overline{P'u_j'})_{,i} + \overline{P'u_{i,j}'} + \overline{P'u_{j,i}'}$$

$$+ \nu(\overline{u_i'u_j'})^{,\ell}{}_{,\ell} - 2\nu\ \overline{u_{i,\ell}'u_j'{}^{,\ell}} \tag{13}$$

An attempt to find an equation for $\overline{u_i'u_j'}$ has introduced several more unknowns, a triple correlation, correlations involving P' and a correlation of derivatives of the velocity fluctuations. Second-order closure consists of devising models for these terms in such a way that the new equation and equations (8) and (9) form a closed set.

In order to ensure applicability over a wide range of circumstances, the models used are invariant under Galilean transformations and under coordinate transformations, they respect symmetry and, of course, they are dimensionally consistent. The "realizability" conditions recently pointed out by Schumann (ref. 2) have not been checked for these models except partially and incidentally. It appears from Schumann's results that a detailed check might provide some useful constraints on the values of the parameters.

Two quantities, Λ and q , appear in most of the models. The determination of the first of these, Λ , the scale, is the motivation for this whole study. The square of the other, q , is defined by

$$q^2 = \overline{u'^{\ell} u'_{\ell}}$$

so it is a measure of the energy of the turbulence.

Most of the modeling constants are designated by rather cumbersome three-or four-letter names suitable for direct use in FORTRAN. The particular names used here were arrived at through an attempt to systematize the naming of the much larger set of constants needed in the compressible case (ref. 3).

The model for triple correlations is practically the same as that given in reference 4:

$$\overline{u'_i u'_j u'_k} = -\ \mathrm{VUU}\ q\Lambda \left((\overline{u'_i u'_j})_{,k} + (\overline{u'_j u'_k})_{,i} + (\overline{u'_k u'_i})_{,j} \right) \tag{14}$$

Note that the three terms are needed for symmetry.

The model for the dissipation term given in reference (4) is

$$\overline{u'_{i,\ell} u'^{,\ell}_j} = \frac{\overline{u'_i u'_j}}{\lambda^2} \tag{15}$$

where λ is another scale given by

$$\lambda^2 = \frac{\Lambda^2}{a + b \frac{q\Lambda}{\nu}}$$

(a and b being constants). Thus, equation (15) can be written

$$\overline{u'_{i,\ell} u'^{,\ell}_j} = \frac{a}{\Lambda^2} \overline{u'_i u'_j} + \frac{bq}{\Lambda\nu} \overline{u'_i u'_j} \tag{16}$$

As was mentioned in reference 4, an isotropic dissipation model in which $g_{ij} q^2/3$ is substituted for $\overline{u'_i u'_j}$ in equations (15) and (16), had been tried and found to lead to negative values of autocorrelations in some circumstances. (That is, we had violated one of the realizability conditions.) Eventually it was realized that the arguments favoring isotropic dissipation apply only at high values of

$$Re_\Lambda = \frac{q\Lambda}{\nu} \tag{17}$$

and that the problems had occurred only at low values, so $g_{ij}q^2/3$ was substituted for $\overline{u_i'u_j'}$ in the second term on the right in equation (16) but not in the first. Actually, the formula in use allows a choice through another parameter, β :

$$\overline{u'_{i,\ell}u'^{,\ell}_j} = \frac{a}{\Lambda^2}\,\overline{u_i'u_j'} + \frac{bq}{\Lambda\nu}\left(\beta\,\overline{u_i'u_j'} + \frac{1-\beta}{3}\,g_{ij}q^2\right) \tag{18}$$

Thus, $\beta = 1$ corresponds to equation (15) or (16) and $\beta = 0$ to isotropic dissipation in the high Reynolds number term only.

The models involving the pressure fluctuation have been expanded considerably from those given in reference 4. They have been suggested by the following considerations. If the divergence of equation (11) is taken, many terms drop out by virtue of equation (10). Solving the resulting expression for the Laplacian of P' and using equation (8) gives

$$P'_{,n}{}^{,n} = -2\bar{u}^{\ell}{}_{,n}u'^{n}{}_{,\ell} - u'^{\ell}{}_{,n}u'^{n}{}_{,\ell} + (\overline{u'^{\ell}u'^{n}})_{,\ell n} \tag{19}$$

Regarding this as Poisson's equation for P' , the solution can be represented, in the absence of wall effects, as an integral

$$P'(\vec{x}) = -\frac{1}{4\pi}\iiint \frac{S'(\vec{\xi})}{|\vec{x}-\vec{\xi}|}\,d\vec{\xi}$$

where S' represents the right-hand side of equation (19). The correlation of P' with another fluctuation, say ϕ' , can thus be written

$$\overline{\phi'P'} = -\frac{1}{4\pi}\iiint \frac{\overline{\phi'(\vec{x})\,S'(\vec{\xi})}}{|\vec{x}-\vec{\xi}|}\,d\vec{\xi}$$

Recognizing that the two-point correlation under the integral can be expected to be negligible outside a region of characteristic

size Λ, and that the value of S' at $\vec{\xi} = \vec{x}$ can be assumed to be representative, this expression is written

$$\overline{\phi' P'} = C\Lambda^2 \overline{\phi' S'} = C\Lambda^2 \left(2\bar{u}^{\ell}{}_{,n}\, \overline{\phi' u'^{n}{}_{,\ell}} + \overline{\phi' u'^{\ell}{}_{,n} u'^{n}{}_{,\ell}} \right)$$

where C is a dimensionless quantity of order one. The P' correlations needed for equation (13) are, therefore, written

$$\overline{P' u_i'} = \Lambda^2 \left(PMU\ \bar{u}^{\ell}{}_{,n}\, \overline{u_i' u'^{n}{}_{,\ell}} + PMU2\ \overline{u_i' u'^{\ell}{}_{,n} u'^{n}{}_{,\ell}} \right) \tag{20}$$

$$\overline{P' u_{i,j}'} = \Lambda^2 \left(PGU\ \bar{u}^{\ell}{}_{,n}\, \overline{u_{i,j}' u'^{n}{}_{,\ell}} + PGU2\ \overline{u_{i,j}' u'^{\ell}{}_{,n} u'^{n}{}_{,\ell}} \right) \tag{21}$$

The four constants introduced here should, by the arguments just given, be all equal to C or $2C$. They have been given separate names in the interest of flexibility and in particular to allow studies of the effect of separate terms.

It appears that not much progress has been made; in modeling two correlations, four new ones that need modeling have been introduced. However, the nature of some of the new ones is such that the modeling is quite constrained. This is particularly true of $\overline{u_i' u_{j,k}'}$. When i and j are equal

$$\overline{u_i' u_{i,k}'} = \frac{1}{2} (\overline{u_i' u_i'})_{,k} \qquad \text{(not summed)}$$

must hold. On the other hand, if j and k are contracted,

$$\overline{u_i' u'^{\ell}{}_{,\ell}} = 0$$

by equation (10). Putting these conditions together practically forces

$$\overline{u_i' u_{j,k}'} = \frac{1}{2} (\overline{u_i' u_j'})_{,k} - \frac{1}{4} \left[g_{jk} (\overline{u'^{\ell} u_i'})_{,\ell} - g_{ik} (\overline{u'^{\ell} u_j'})_{,\ell} \right] \tag{22}$$

a model remarkable for having no adjustable constants.

Similar considerations lead to

$$\overline{u'_{i,j}u'_{n,m}} = \frac{WGU}{\Lambda^2}\left(g_{mj}\overline{u'_iu'_n} - \frac{1}{3}\,g_{nm}\overline{u'_iu'_j} - \frac{1}{3}\,g_{ij}\overline{u'_nu'_m} + \frac{1}{9}\,g_{ij}g_{nm}\;q^2\right)$$

$$+ \frac{WGUB}{\Lambda^2}\left(g_{mj}\overline{u'_iu'_n} - g_{ni}\overline{u'_ju'_m}\right)$$

$$+ \frac{WGUC}{\Lambda^2}\left(g_{mi}g_{nj}\;q^2 - \frac{1}{3}\,g_{ij}g_{mn}\;q^2\right) \qquad (23)$$

The next has just one free index, so it is modeled by

$$\overline{u'_iu'^{\ell}{}_{,n}u'^{n}{}_{,\ell}} = WWU1\;\frac{q}{\Lambda}\left(\overline{u'^{\ell}u'_i}\right)_{,\ell} \qquad (24)$$

Finally, the model

$$\overline{u'_{i,j}u'^{\ell}{}_{,n}u'^{n}{}_{,\ell}} = WWGU\;\frac{q}{\Lambda^3}\left(\overline{u'_iu'_j} - \frac{1}{3}\,g_{ij}\;q^2\right) \qquad (25)$$

is governed by the requirement that it be zero when i and j are contracted.

Substitution of equations (22) through (25) into (20) and (21) with the use of equation (8) gives

$$\overline{P'u'_i} = PMU\;\Lambda^2\left(\frac{1}{2}\,\bar{u}^{\ell}{}_{,n}\left(\overline{u'_iu'^{n}}\right)_{,\ell} + \frac{1}{4}\,\bar{u}_{i,n}\left(\overline{u'^{\ell}u'^{n}}\right)_{,\ell}\right)$$

$$+ PUW\;q\Lambda\left(\overline{u'^{\ell}u'_i}\right)_{,\ell} \qquad (26)$$

$$\overline{P'u'_{i,j}} = GUA\left(\bar{u}_{j,n}\;\overline{u'_iu'^{n}} - \frac{1}{3}\,g_{ij}\bar{u}^{\ell}{}_{,n}\;\overline{u'_{\ell}u'^{n}}\right)$$

$$+ GUB\left(\bar{u}_{j,n}\;\overline{u'_iu'^{n}} - \bar{u}^{\ell}{}_{,i}\;\overline{u'_ju'_{\ell}}\right) + GUC\;\bar{u}_{i,j}\;q^2$$

$$+ \frac{1}{2}\;GU2\;\frac{q}{\Lambda}\left(\overline{u'_iu'_j} - \frac{1}{3}\,g_{ij}\;q^2\right) \qquad (27)$$

where $PUW = PMU2 \cdot WWU1$

$GUA = PGU \cdot WGU$

$GUB = PGU \cdot WGUB$

$GUC = PGU \cdot WGUC$

$GU2 = 2 \cdot PGU2 \cdot WWGU$

Equations (26) and (27) are taken to be models. The steps leading to them are considered to be merely suggestive. The last term in each is essentially the corresponding model in reference 4. The GUA, GUB, and GUC terms in equation (27) are equivalent to terms arrived at by a different route by Naot, Shavit, and Wolfshtein (ref. 5). Their parameters, α, β, and γ are related to these by

$$\alpha = GUA + GUB$$

$$\beta = GUB$$

$$\gamma = 3 \cdot GUC$$

(It should be pointed out that the relationship between α, β, and γ which is given in reference 5 is rescinded in reference 6.)

One reason for not taking literally the process leading to equations (26) and (27) is that equation (23) does not agree with (18) when j and m are contracted. The fact that the latter is used in a term multiplied by the viscosity serves as one justification for the discrepancy. The use of equation (23) only as a suggestion for arriving at (27) is another.

The substitution of the model given by equations (14), (18), (26), and (27) into (13) yields the modeled equation for $\overline{u_i' u_j'}$:

$$(\overline{u_i'u_j'})_{t} + \bar{u}^{\ell}(\overline{u_i'u_j'})_{,\ell} + \overline{u'^{\ell}u_i'}\ \bar{u}_{j,\ell} + \overline{u'^{\ell}u_j'}\ \bar{u}_{i,\ell}$$

$$- VUU\left[q\Lambda\left((\overline{u_i'u_j'})^{,\ell} + (\overline{u'^{\ell}u_i'})_{,j} + (\overline{u'^{\ell}u_j'})_{,i}\right)\right]_{,\ell} =$$

$$= - PMU\left[\Lambda^2\left(\frac{1}{2}\,\bar{u}^{\ell}{}_{,n}(\overline{u_i'u'^{n}})_{,\ell} + \frac{1}{4}\,\bar{u}_{i,n}(\overline{u'^{\ell}u'^{n}})_{,\ell}\right)\right]_{,j} +$$

$$- \text{PMU}\left[\Lambda^2\left(\frac{1}{2}\,\bar{u}^{\ell}{}_{,n}\left(\overline{u_j'u'^n}\right)_{,\ell} + \frac{1}{4}\,\bar{u}_{j,n}\left(\overline{u'^{\ell}u'^n}\right)_{,\ell}\right)\right]_{,i}$$

$$- \text{PUW}\left[\left(q\Lambda\ \left(\overline{u'^{\ell}u_i'}\right)_{,\ell}\right)_{,j} + \left(q\Lambda\ \left(\overline{u'^{\ell}u_j'}\right)_{,\ell}\right)_{,i}\right]$$

$$+ \text{GUA}\left(\bar{u}_{j,n}\ \overline{u_i'u'^n} + \bar{u}_{i,n}\ \overline{u_j'u'^n} - \frac{2}{3}\,g_{ij}\bar{u}^{\ell}{}_{,n}\ \overline{u_{\ell}'u'^n}\right)$$

$$+ \text{GUB}\left[\left(\bar{u}_{j,n} - \bar{u}_{n,j}\right)\ \overline{u_i'u'^n} + \left(\bar{u}_{i,n} - \bar{u}_{n,i}\right)\overline{u_j'u'^n}\right]$$

$$+ \text{GUC}\ \left(\bar{u}_{i,j} + \bar{u}_{j,i}\right)q^2$$

$$+ \text{GU2}\ \frac{q}{\Lambda}\left(\overline{u_i'u_j'} - \frac{1}{3}\,g_{ij}\ q^2\right)$$

$$+ \nu\ \left(\overline{u_i'u_j'}\right)^{,\ell}{}_{,\ell} - 2\ \frac{\nu a}{\Lambda^2}\ \overline{u_i'u_j'}$$

$$- 2b\ \frac{q}{\Lambda}\left(\beta\ \overline{u_i'u_j'} + \frac{1-\beta}{3}\,g_{ij}\ q^2\right) \tag{28}$$

Given a method for determining Λ , equations (8), (9), and (28) form a closed set.

3.- THE SCALE TENSOR

Understanding of Λ is sought through the scale tensor, and the scale tensor is defined in terms of the two-point velocity correlation. Consideration of two points a finite distance apart is cumbersome in general coordinates. Therefore, Cartesian coordinates are employed for most of this section.

Consider two points in the same flow: $x = (x^1, x^2, x^3)$ and $\xi = (\xi^1, \xi^2, \xi^3)$. Proceeding much as in the development of equation (12), rewrite equation (11) with j substituted for i, interpret it as applying at ξ, and multiply by $u_i'(x)$. Add the result to the original equation (11) interpreted as applying at x and multiplied by $u_j'(\xi)$. Note that a function of x behaves as a constant in differentiation with respect to ξ and vice versa. Using equations (8) and (10) and averaging, the result is

$$\begin{aligned}
&\frac{\partial \overline{u_i'(x)u_j'(\xi)}}{\partial t} + \bar{u}^\ell(\xi)\frac{\partial \overline{u_i'(x)u_j'(\xi)}}{\partial \xi^\ell} + \bar{u}^\ell(x)\frac{\partial \overline{u_i'(x)u_j'(\xi)}}{\partial x^\ell} \\
&\quad + \overline{u_i'(x)u'^\ell(\xi)}\frac{\partial \bar{u}_j(\xi)}{\partial \xi^\ell} + \overline{u'^\ell(x)u_j'(\xi)}\frac{\partial \bar{u}_i(x)}{\partial x^\ell} \\
&\quad + \frac{\partial \overline{u'^\ell(\xi)u_i'(x)u_j'(\xi)}}{\partial \xi^\ell} + \frac{\partial \overline{u'^\ell(x)u'_i(x)u_j'(\xi)}}{\partial x^\ell} = \\
&\quad = -\frac{\partial \overline{P'(\xi)u_i'(x)}}{\partial \xi^j} - \frac{\partial \overline{P'(x)u_j'(\xi)}}{\partial x^i} \\
&\quad + \nu\left(\frac{\partial^2 \overline{u_i'(x)u_j'(\xi)}}{\partial \xi^\ell \partial \xi^\ell} + \frac{\partial^2 \overline{u_i'(x)u_j'(\xi)}}{\partial x^\ell \partial x^\ell}\right)
\end{aligned} \tag{29}$$

It is not obvious that this equation is equivalent to equation (12) in the limit as x and ξ approach the same point. The demonstration can be accomplished using techniques that are employed here for a different purpose.

Sandri introduced the concept of modeling at this stage (ref. 7). One advantage is that the modeled equation should reduce to equation (28) as x and y approach each other. This helps establish the form of the models and also determines some of the constants. Thus, part of the arbitrariness in other approaches (see, for example, ref. 8) is avoided.

Another advantage is that it is possible to relate the properties of the modeled equation to the spectral properties of the turbulence of which so much is known. This aspect is not, however, pursued in this paper except to note that it has influenced the models. A third advantage is that the terms to be modeled have known properties that can be exploited in the construction of the model.

Some technical apparatus is needed for what follows. New independent variables, $\zeta = (\zeta^1, \zeta^2, \zeta^3)$, the centroid, and $r = (r^1, r^2, r^3)$, the relative separation, are defined by

$$\zeta = \frac{1}{2}(x + \xi) \tag{30}$$

$$r = \xi - x \tag{31}$$

Note that

$$\frac{\partial}{\partial x^i} = \frac{1}{2}\frac{\partial}{\partial \zeta^i} - \frac{\partial}{\partial r^i} \tag{32}$$

$$\frac{\partial}{\partial \xi^i} = \frac{1}{2}\frac{\partial}{\partial \zeta^i} + \frac{\partial}{\partial r^i} \tag{33}$$

$$\frac{\partial}{\partial \zeta^i} = \frac{\partial}{\partial x^i} + \frac{\partial}{\partial \xi^i} \tag{34}$$

$$\frac{\partial}{\partial r^i} = -\frac{1}{2}\frac{\partial}{\partial x^i} + \frac{1}{2}\frac{\partial}{\partial \xi^i} \tag{35}$$

A consequence of equation (34) is that for a function of x only, say θ

$$\frac{\partial \theta}{\partial x^i} = \frac{\partial \theta}{\partial \zeta^i} \quad , \quad \theta = \theta(x) \tag{36}$$

and for a function of ξ only, say χ

$$\frac{\partial \chi}{\partial \xi^i} = \frac{\partial \chi}{\partial \zeta^i} \quad , \quad \chi = \chi(\xi) \tag{37}$$

Another new symbol is

$$R_{ij} = R_{ij}(\zeta,r) = \overline{u'_i(x)u'_j(\xi)} \tag{38}$$

Note that

$$R_{ij}(\zeta,r) = R_{ji}(\zeta,-r)$$

but in general it has neither the $i - j$ symmetry nor the r symmetry separately. A new tensor, $S_{ij}(\zeta,r)$, a functional of R_{ij}, is introduced. Its exact definition is not needed for this paper; it is enough to record the following two properties:

I. S_{ij} is isotropic as a function of r (ref. 9).

II. If R_{ij} itself is isotropic, then $S_{ij} = R_{ij}$.

It can be shown from Property I that

$$S_{ij}(\zeta,0) = \frac{1}{3} g_{ij} S^{\ell}{}_{\ell}(\zeta,0) \tag{39}$$

and

$$\iiint \frac{S_{ij}(\zeta,r)}{4\pi|r|^2}\, dr^1 dr^2 dr^3 = \frac{1}{3} g_{ij} \iiint \frac{S^{\ell}{}_{\ell}(\zeta,r)}{4\pi|r|^2}\, dr^1 dr^2 dr^3 \tag{40}$$

Another auxiliary tensor, N_{ij}, is required to have the following two properties:

I. $$N_{ij}(\zeta,0) = R_{ij}(\zeta,0) = \overline{u'_i u'_j} \tag{41}$$

II. $$\iiint \frac{N_{ij}(\zeta,r)}{4\pi|r|^2}\, dr^1 dr^2 dr^3 = 0 \tag{42}$$

Otherwise, for the purposes of this paper, N_{ij} is arbitrary and can be different in the different places it appears in the models below. An example is

$$N_{ij} = R_{ij} + r^{\ell} \frac{\partial R_{ij}}{\partial r^{\ell}}$$

The terms of equation (29) are now considered in groups. The first three, in terms of R_{ij}, are

$$\frac{\partial R_{ij}}{\partial t} + \bar{u}^{\ell}(\xi) \frac{\partial R_{ij}}{\partial \xi^{\ell}} + \bar{u}^{\ell}(x) \frac{\partial R_{ij}}{\partial x^{\ell}} =$$

$$= \frac{\partial R_{ij}}{\partial t} + \frac{1}{2}\left(\bar{u}^{\ell}(\xi) + \bar{u}^{\ell}(x)\right) \frac{\partial R_{ij}}{\partial \zeta^{\ell}} + \left(\bar{u}^{\ell}(\xi) - \bar{u}^{\ell}(x)\right) \frac{\partial R_{ij}}{\partial r^{\ell}} \tag{43}$$

by equations (32) and (33). On the right-hand side of (43) and in similar circumstances below, ζ and r are the independent variables; x and ξ are merely abbreviations for $\zeta - \frac{1}{2}r$ and $\zeta + \frac{1}{2}r$, respectively. The next two terms, the "production terms," are

$$R_i{}^{\ell} \frac{\partial \bar{u}_j(\xi)}{\partial \xi^{\ell}} + R^{\ell}{}_j \frac{\partial \bar{u}_i(x)}{\partial x^{\ell}} = R_i{}^{\ell} \frac{\partial \bar{u}_j(\xi)}{\partial \zeta^{\ell}} + R^{\ell}{}_j \frac{\partial \bar{u}_i(x)}{\partial \zeta^{\ell}} \tag{44}$$

by equations (36) and (37).

The triple correlation terms are modeled:

$$\frac{\partial \overline{u'^{\ell}(\xi) u_i'(x) u_j'(\xi)}}{\partial \xi^{\ell}} + \frac{\partial \overline{u'^{\ell}(x) u_i'(x) u_j'(\xi)}}{\partial x^{\ell}} =$$

$$= - \text{VUU} \frac{\partial}{\partial \zeta^{\ell}} \left[q\Lambda \left(\frac{\partial R_{ij}}{\partial \zeta^{\ell}} + \frac{\partial R_i{}^{\ell}}{\partial \zeta^{j}} + \frac{\partial R^{\ell}{}_j}{\partial \zeta^{i}} \right) \right]$$

$$+ \frac{2\nu \text{ATC}}{\Lambda^2} (R_{ij} - N_{ij})$$

$$+ 2\text{BTC} \frac{q}{\Lambda} \left[\beta (R_{ij} - N_{ij}) + \frac{1-\beta}{3} g_{ij} \left(R^{\ell}{}_{\ell} - N^{\ell}{}_{\ell} \right) \right] \tag{45}$$

(Here and in what follows, q and Λ and other functions of a single vector variable are to be evaluated at ζ if not otherwise indicated.) Thus, it is hypothesized that these terms have a diffusive effect and also a pseudo-dissipative effect — "pseudo" since the last terms tend to drive R_{ij} toward N_{ij}, or away, depending on the signs of the constants, instead of toward 0.

Turning to the terms involving pressure fluctuations, note that

$$\frac{\partial \overline{P'(\xi)u'_i(x)}}{\partial \xi^j} = \overline{\frac{\partial P'(\xi)}{\partial \xi^j} u'_i(x)} = \overline{\frac{\partial P'(\xi)}{\partial \zeta^j} u'_i(x)} \tag{46}$$

by equation (37). But

$$\frac{\partial \overline{P'(\xi)u'_i(x)}}{\partial \zeta^j} = \overline{\frac{\partial P'(\xi)}{\partial \zeta^j} u'_i(x)} + \overline{P'(\xi) \frac{\partial u'_i(x)}{\partial \zeta^j}} \tag{47}$$

so, using equation (47) with (46) and making similar manipulations with the other P' term,

$$\frac{\partial \overline{P'(\xi)u'_i(x)}}{\partial \xi^j} + \frac{\partial \overline{P'(x)u'_j(\xi)}}{\partial x^i} =$$

$$= \frac{\partial \overline{P'(\xi)u'_i(x)}}{\partial \zeta^j} + \frac{\partial \overline{P'(x)u'_j(\xi)}}{\partial \zeta^i} - \overline{P'(\xi) \frac{\partial u'_i(x)}{\partial \zeta^j}} - \overline{P'(x) \frac{\partial u'_j(\xi)}{\partial \zeta^i}} \tag{48}$$

The first pair is modeled:

$$\frac{\partial \overline{P'(\xi)u'_i(x)}}{\partial \zeta^j} + \frac{\partial \overline{P'(x)u'_j(\xi)}}{\partial \zeta^i} =$$

$$= \mathrm{PMU}\, \frac{\partial}{\partial \zeta^j}\left[\Lambda^2\left(\frac{1}{2}\, \frac{\partial \bar{u}^\ell(x)}{\partial \zeta^n}\, \frac{\partial R_i{}^n}{\partial \zeta^\ell} + \frac{1}{4}\, \frac{\partial \bar{u}_i(x)}{\partial \zeta^n}\, \frac{\partial R^{\ell n}}{\partial \zeta^\ell}\right)\right]$$

$$+ \mathrm{PMU}\, \frac{\partial}{\partial \zeta^i}\left[\Lambda^2\left(\frac{1}{2}\, \frac{\partial \bar{u}^\ell(\xi)}{\partial \zeta^n}\, \frac{\partial R^n{}_j}{\partial \zeta^\ell} + \frac{1}{4}\, \frac{\partial \bar{u}_j(\xi)}{\partial \zeta^n}\, \frac{\partial R^{n\ell}}{\partial \zeta^\ell}\right)\right] +$$

$$+ \text{PUW}\left[\frac{\partial}{\partial\zeta^j}\left(q\Lambda\,\frac{\partial R_i^{\ \ell}}{\partial\zeta^\ell}\right) + \frac{\partial}{\partial\zeta^i}\left(q\Lambda\,\frac{\partial R^\ell_{\ j}}{\partial\zeta^\ell}\right)\right]$$

$$- \text{POC}\,\frac{q}{\Lambda}\,\frac{\partial\Lambda}{\partial\zeta^\ell}\,\frac{\partial\Lambda}{\partial\zeta^\ell}\,(R_{ij} - N_{ij}) \tag{49}$$

Here, in addition to the two types of diffusion terms found in equation (28), another term is included. Again, this term tends to drive R_{ij} toward (or away from) N_{ij} but only where there is a gradient in Λ. It is known (ref. 9) that $\overline{P'(\xi)u_i'(x)}$ is zero in isotropic turbulence. Therefore, all the terms on the right-hand side in equation (49) must be zero in that situation; in the last term that is assured by the factor $(\partial\Lambda/\partial\zeta^\ell)\cdot(\partial\Lambda/\partial\zeta^\ell)$.

The other pair of terms on the right-hand side of equation (48) is modeled:

$$\overline{P'(\xi)\,\frac{\partial u_i'(x)}{\partial\zeta^j}} + \overline{P'(x)\,\frac{\partial u_j'(\xi)}{\partial\zeta^i}} =$$

$$= \text{GUA}\left[\frac{\partial\bar{u}_j(\xi)}{\partial\zeta^n}\,R_i^{\ n} + \frac{\partial\bar{u}_i(x)}{\partial\zeta^n}\,R^n_{\ j}\right.$$

$$\left. - \frac{1}{3}\,g_{ij}\left(\frac{\partial\bar{u}^\ell(\xi)}{\partial\zeta^n}\,R_\ell^{\ n} + \frac{\partial\bar{u}^\ell(x)}{\partial\zeta^n}\,R^n_{\ \ell}\right)\right]$$

$$+ \text{GUB}\left[\left(\frac{\partial\bar{u}_j(\xi)}{\partial\zeta^n} - \frac{\partial\bar{u}_n(\xi)}{\partial\zeta^j}\right)R_i^{\ n} + \left(\frac{\partial\bar{u}_i(x)}{\partial\zeta^n} - \frac{\partial\bar{u}_n(x)}{\partial\zeta^i}\right)R^n_{\ j}\right]$$

$$+ \text{GUC}\left(\frac{\partial\bar{u}_j(\xi)}{\partial\zeta^i} + \frac{\partial\bar{u}_i(x)}{\partial\zeta^j}\right)R^n_{\ n}$$

$$+ \text{GU2}\,\frac{q}{\Lambda}\left[R_{ij} - S_{ij} - \frac{1}{3}\,g_{ij}\left(R^n_{\ n} - S^n_{\ n}\right)\right] \tag{50}$$

These too must all go to zero for isotropic turbulence since the left-hand side of equation (48) is zero in that case. That is the reason for the introduction of S_{ij} in the last term (according to

Property II). The other terms satisfy the requirement by virtue of the fact that homogeneous turbulence will not exist in the presence of gradients of the mean velocity.

For the final terms in equation (29), observe that

$$\frac{\partial^2 \overline{u_i'(x) u_j'(\xi)}}{\partial\zeta^\ell \partial\zeta^\ell} = \overline{\frac{\partial^2 u_i'(x)}{\partial\zeta^\ell \partial\zeta^\ell} u_j'(\xi)} + 2\, \overline{\frac{\partial u_i'(x)}{\partial\zeta^\ell} \frac{\partial u_j'(\xi)}{\partial\zeta^\ell}} + \overline{u_i'(x) \frac{\partial^2 u_j'(\xi)}{\partial\zeta^\ell \partial\zeta^\ell}}$$

or, using equations (36) and (37)

$$\frac{\partial^2 \overline{u_i'(x) u_j'(\xi)}}{\partial\zeta^\ell \partial\zeta^\ell} = \frac{\partial^2 \overline{u_i'(x) u_j'(\xi)}}{\partial x^\ell \partial x^\ell} + 2\, \overline{\frac{\partial u_i'(x)}{\partial\zeta^\ell} \frac{\partial u_j'(\xi)}{\partial\zeta^\ell}} + \frac{\partial^2 \overline{u_i'(x) u_j'(\xi)}}{\partial\xi^\ell \partial\xi^\ell}$$

Thus, the last two terms of equation (29) can be written

$$\nu \left(\frac{\partial^2 \overline{u_i'(x) u_j'(\xi)}}{\partial\xi^\ell \partial\xi^\ell} + \frac{\partial^2 \overline{u_i'(x) u_j'(\xi)}}{\partial x^\ell \partial x^\ell} \right) = \nu \frac{\partial^2 R_{ij}}{\partial\zeta^\ell \partial\zeta^\ell} - 2\nu\, \overline{\frac{\partial u_i'(x)}{\partial\zeta^\ell} \frac{\partial u_j'(\xi)}{\partial\zeta^\ell}} \qquad (51)$$

One more model is needed:

$$\overline{\frac{\partial u_i'(x)}{\partial\zeta^\ell} \frac{\partial u_j'(\xi)}{\partial\zeta^\ell}} = \frac{a}{\Lambda^2} R_{ij} + \frac{bq}{\Lambda\nu} \left(\beta R_{ij} + \frac{1-\beta}{3} g_{ij} R^\ell{}_\ell \right) \qquad (52)$$

Putting together the results of equations (43) through (52), equation (29) becomes the modeled two-point correlation equation:

$$\frac{\partial R_{ij}}{\partial t} + \frac{1}{2}\left(\bar{u}^\ell(\xi) + \bar{u}^\ell(x)\right) \frac{\partial R_{ij}}{\partial\zeta^\ell} + \left(\bar{u}^\ell(\xi) - \bar{u}^\ell(x)\right) \frac{\partial R_{ij}}{\partial r^\ell}$$

$$+ R_i{}^\ell \frac{\partial \bar{u}_j(\xi)}{\partial\zeta^\ell} + R^\ell{}_j \frac{\partial \bar{u}_i(x)}{\partial\zeta^\ell} - \mathrm{VUU} \frac{\partial}{\partial\zeta^\ell}\left[q\Lambda \left(\frac{\partial R_{ij}}{\partial\zeta^\ell} + \frac{\partial R_i{}^\ell}{\partial\zeta^j} + \frac{\partial R^\ell{}_j}{\partial\zeta^i} \right)\right]$$

$$+ \frac{2\nu \mathrm{ATC}}{\Lambda^2} (R_{ij} - N_{ij}) + 2\mathrm{BTC}\, \frac{q}{\Lambda}\left[\beta(R_{ij} - N_{ij}) + \frac{1-\beta}{3} g_{ij} (R^\ell{}_\ell - N^\ell{}_\ell) \right] =$$

$$
\begin{aligned}
= &- \mathrm{PMU}\,\frac{\partial}{\partial\zeta^j}\left[\Lambda^2\left(\frac{1}{2}\,\frac{\partial\bar{u}^\ell(x)}{\partial\zeta^n}\,\frac{\partial R_i{}^n}{\partial\zeta^\ell} + \frac{1}{4}\,\frac{\partial\bar{u}_i(x)}{\partial\zeta^n}\,\frac{\partial R^{\ell n}}{\partial\zeta^\ell}\right)\right] \\
&- \mathrm{PMU}\,\frac{\partial}{\partial\zeta^i}\left[\Lambda^2\left(\frac{1}{2}\,\frac{\partial\bar{u}^\ell(\xi)}{\partial\zeta^n}\,\frac{\partial R^n{}_j}{\partial\zeta^\ell} + \frac{1}{4}\,\frac{\partial\bar{u}_j(\xi)}{\partial\zeta^n}\,\frac{\partial R^{n\ell}}{\partial\zeta^\ell}\right)\right] \\
&- \mathrm{PUW}\left[\frac{\partial}{\partial\zeta^j}\left(q\Lambda\,\frac{\partial R_i{}^\ell}{\partial\zeta^\ell}\right) + \frac{\partial}{\partial\zeta^i}\left(q\Lambda\,\frac{\partial R^\ell{}_j}{\partial\zeta^\ell}\right)\right] \\
&+ \mathrm{POC}\,\frac{q}{\Lambda}\,\frac{\partial\Lambda}{\partial\zeta^\ell}\,\frac{\partial\Lambda}{\partial\zeta^\ell}\,(R_{ij} - N_{ij}) \\
&+ \mathrm{GUA}\left[\frac{\partial\bar{u}_j(\xi)}{\partial\zeta^n}\,R_i{}^n + \frac{\partial\bar{u}_i(x)}{\partial\zeta^n}\,R^n{}_j \right. \\
&\qquad\qquad \left. - \frac{1}{3}\,g_{ij}\left(\frac{\partial\bar{u}^\ell(\xi)}{\partial\zeta^n}\,R_\ell{}^n + \frac{\partial\bar{u}^\ell(x)}{\partial\zeta^n}\,R^n{}_\ell\right)\right] \\
&+ \mathrm{GUB}\left[\left(\frac{\partial\bar{u}_j(\xi)}{\partial\zeta^n} - \frac{\partial\bar{u}_n(\xi)}{\partial\zeta^j}\right)R_i{}^n + \left(\frac{\partial\bar{u}_i(x)}{\partial\zeta^n} - \frac{\partial\bar{u}_n(x)}{\partial\zeta^i}\right)R^n{}_j\right] \\
&+ \mathrm{GUC}\left(\frac{\partial\bar{u}_j(\xi)}{\partial\zeta^i} + \frac{\partial\bar{u}_i(x)}{\partial\zeta^j}\right)R^n{}_n + \mathrm{GU2}\,\frac{q}{\Lambda}\left[R_{ij} - S_{ij} - \frac{1}{3}\,q_{ij}(R^n{}_n - S^n{}_n)\right] \\
&+ \nu\,\frac{\partial^2 R_{ij}}{\partial\zeta^\ell\partial\zeta^\ell} - 2\,\frac{\nu a}{\Lambda^2}\,R_{ij} - 2b\,\frac{q}{\Lambda}\left(\beta R_{ij} + \frac{1-\beta}{3}\,g_{ij}\,R^\ell{}_\ell\right) \qquad (53)
\end{aligned}
$$

It is easy to show that equation (53) reduces to equation (28) as r goes to zero. The terms containing N_{ij} drop out by equation (41) and S_{ij} drops out by equation (39). The term $\left(\bar{u}^\ell(\xi) - \bar{u}^\ell(x)\right)\left(\partial R_{ij}/\partial r^\ell\right)$ becomes zero since ξ and x both approach ζ. The rest reduces term by term to equation (28).

The scale tensor, Λ_{ij}, is defined through the quantity

$$
\Omega_{ij} = \iiint \frac{R_{ij}(\zeta, r)}{4\pi|r|^2}\,dr^1\,dr^2\,dr^3 \qquad (54)
$$

(It is seen that Ω_{ij} is symmetrical in i and j .) Then,

$$\Lambda_{ij} = \frac{\Omega_{ij}}{q^2} \tag{55}$$

Actually, most of the analysis is done in terms of Ω_{ij} . Accordingly, equation (53) is divided through by $4\pi|r|^2$ and integrated over all r space. Many of the terms are expressible immediately in terms of Ω_{ij} . For example,

$$\iiint PUW \frac{\partial}{\partial\zeta^j}\left(q\Lambda \frac{\partial R_i{}^\ell}{\partial\zeta^\ell}\right)\frac{dr^1 dr^2 dr^3}{4\pi|r|^2} =$$

$$= PUW \frac{\partial}{\partial\zeta^j}\left[q\Lambda \frac{\partial}{\partial\zeta^\ell}\iiint \frac{R_i{}^\ell dr^1 dr^2 dr^3}{4\pi|r|^2}\right] =$$

$$= PUW \frac{\partial}{\partial\zeta^j}\left(q\Lambda \frac{\partial\Omega_i{}^\ell}{\partial\zeta^\ell}\right)$$

since ζ and r are independent and q and Λ are, by the convention adopted above, functions of ζ only. It is seen by equation (40) that S_{ij} drops out and by equation (42) that N_{ij} drops out. This is in contrast with the limit as r goes to zero where the whole terms containing N_{ij} dropped out.

Terms containing factors evaluated at x or ξ need an additional assumption before the integration can be completed. Roughly speaking, the assumption is that the variation of R_{ij} with ζ is smoother than the variation with r , that is, the flow is locally homogeneous. More formally, the assumption is that it is sufficient to use the leading term in a moment expansion. Explicitly, it is assumed that

$$R_{ij}(\zeta,r) = M_{ij}(\zeta)4\pi|r|^2\ \delta(r) \tag{56}$$

where $\delta(r)$ is the three-dimensional Dirac delta function defined by

$$\iiint \phi(r)\ \delta(r)\ dr^1\ dr^2\ dr^3 = \phi(0) \tag{57}$$

for any continuous function ϕ .

Substitution of equation (56) into (54) shows, with the help of (57) that

$$M_{ij} = \Omega_{ij}$$

so that equation (56) can be rewritten

$$R_{ij} = \Omega_{ij}\, 4\pi |r|^2\, \delta(r)$$

With this substitution, the rest of the integrals can be performed. This has the effect, with one exception, of changing the point of evaluation of the coefficients from x or ξ to ζ, by equation (57). The exception is the term

$$Q_{ij} = \iiint \left(\bar{u}^\ell(\xi) - \bar{u}^\ell(x)\right) \frac{\partial\left(\Omega_{ij}\; 4\pi|r|^2\delta(r)\right)}{\partial r^\ell}\; \frac{dr^1 dr^2 dr^3}{4\pi|r|^2}$$

This turns out to be zero, but it takes a few steps to show that.

Consider more generally,

$$\Phi = \iiint \left(\phi(\xi) - \phi(x)\right) \frac{\partial}{\partial r^\ell} \left(|r|^2 \delta(r)\right) \frac{dr^1 dr^2 dr^3}{|r|^2}$$

Assume ϕ can be expanded in Taylor's series. Then

$$\phi(\xi) - \phi(x) = \phi\left(\zeta + \frac{1}{2} r\right) - \phi\left(\zeta - \frac{1}{2} r\right) = r^n \frac{\partial \phi(\zeta)}{\partial r^n} + \ldots$$

(If more terms were carried, their contribution would easily be shown to be zero.) Now,

$$\frac{\partial |r|^2}{\partial r^\ell} = 2r^\ell$$

so

$$\Phi = \iiint \frac{\partial \phi}{\partial r^n} \left(2r^n r^\ell \delta(r) + r^n |r|^2 \frac{\partial \delta(r)}{\partial r^\ell}\right) \frac{dr^1 dr^2 dr^3}{|r|^2}$$

$$\Phi = \frac{\partial\phi}{\partial r^n}\iiint\left(2\,\frac{r^n r^\ell}{|r|^2}\,\delta(r) + r^n\,\frac{\partial\delta(r)}{\partial r^\ell}\right) dr^1\, dr^2\, dr^3$$

$$= \frac{\partial\phi}{\partial r^n}\iiint\left(2\,\frac{r^n r^\ell}{|r|^2}\,\delta(r) - \frac{\partial r^n}{\partial r^\ell}\,\delta(r)\right) dr^1\, dr^2\, dr^3$$

the last step being the result of integration by parts of the second term. Let

$$\psi_{n\ell} = \iiint 2\,\frac{r^n r^\ell}{|r|^2}\,\delta(r)\; dr^1\, dr^2\, dr^3$$

By the symmetries involved, $\psi_{n\ell}$ must be proportional to the Kronecker delta, $\delta_{n\ell}$. On contraction,

$$\psi^\ell{}_\ell = \iiint 2\,\frac{|r|^2}{|r|^2}\,\delta(r)\; dr^1\, dr^2\, dr^3 = 2$$

using equation (57). Therefore,

$$\psi_{n\ell} = \frac{2}{3}\,\delta_{n\ell}$$

since $\delta^\ell{}_\ell = 3$. Also,

$$\frac{\partial r^n}{\partial r^\ell} = \delta_{n\ell}$$

so Φ can be written

$$\Phi = \frac{\partial\phi}{\partial r^n}\left(\frac{2}{3}\,\delta_{n\ell} - \delta_{n\ell}\right) = -\,\frac{1}{3}\,\frac{\partial\phi}{\partial r^\ell}$$

Finally, then,

$$Q_{ij} = -\,\frac{1}{3}\,\frac{\partial\bar{u}^\ell}{\partial r^\ell}\,\Omega_{ij}$$

or, by equations (8) and (35)

$$Q_{ij} = 0$$

The equation for Ω_{ij} can now be written. Since the r dependence has been eliminated by the integration, only one point is involved in the equation so general coordinate notation is employed once more. The contraction (trace) of Ω_{ij} is written

$$\Omega = \Omega^{\ell}{}_{\ell} \tag{58}$$

To take advantage of combinations of terms that occur, write

$$\begin{aligned} AOM &= a + ATC \\ BOM &= b + BTC \end{aligned} \tag{59}$$

The equation is

$$\begin{aligned}
&\Omega_{ij_t} + \bar{u}^{\ell}\, \Omega_{ij,\ell} + \Omega^{\ell}{}_{i}\, \bar{u}_{j,\ell} + \Omega^{\ell}{}_{j}\, \bar{u}_{i,\ell} \\
&\quad - VUU \left[q\Lambda\left(\Omega_{ij}{}^{,\ell} + \Omega^{\ell}{}_{i,j} + \Omega^{\ell}{}_{j,i}\right) \right]_{,\ell} = \\
&\quad = - PMU \left[\Lambda^2 \left(\frac{1}{2}\, \bar{u}^{\ell}{}_{,n}\, \Omega^{n}{}_{i,\ell} + \frac{1}{4}\, \bar{u}_{i,n}\, \Omega^{\ell n}{}_{,\ell} \right) \right]_{,j} \\
&\quad - PMU \left[\Lambda^2 \left(\frac{1}{2}\, \bar{u}^{\ell}{}_{,n}\, \Omega^{n}{}_{j,\ell} + \frac{1}{4}\, \bar{u}_{j,n}\, \Omega^{\ell n}{}_{,\ell} \right) \right]_{,i} \\
&\quad - PUW \left[\left(q\Lambda\, \Omega^{\ell}{}_{i,\ell} \right)_{,j} + \left(q\Lambda\, \Omega^{\ell}{}_{j,\ell} \right)_{,i} \right] \\
&\quad + POC\, \frac{q}{\Lambda}\, \Lambda^{,\ell} \Lambda_{,\ell}\, \Omega_{ij} \\
&\quad + GUA \left(\bar{u}_{j,n}\, \Omega^{n}{}_{i} + \bar{u}_{i,n}\, \Omega^{n}{}_{j} - \frac{2}{3}\, g_{ij}\, \bar{u}^{\ell}{}_{,n}\, \Omega^{n}{}_{\ell} \right) +
\end{aligned}$$

$$+ \mathrm{GUB}\left[\left(\bar{u}_{j,n} - \bar{u}_{n,j}\right)\Omega^{n}{}_{i} + \left(\bar{u}_{i,n} - \bar{u}_{n,i}\right)\Omega^{n}{}_{j}\right]$$

$$+ \mathrm{GUC}\left(\bar{u}_{j,i} + \bar{u}_{i,j}\right)\Omega + \mathrm{GU2}\,\frac{q}{\Lambda}\left(\Omega_{ij} - \frac{1}{3}\,g_{ij}\Omega\right)$$

$$+ \nu\Omega_{ij}{}^{,\ell}{}_{,\ell} - 2\,\frac{\nu\mathrm{AOM}}{\Lambda^2}\,\Omega_{ij} - 2\mathrm{BOM}\,\frac{q}{\Lambda}\left(\beta\,\Omega_{ij} + \frac{1-\beta}{3}\,g_{ij}\Omega\right) \qquad (60)$$

From this equation an equation for Λ_{ij} can be derived, as follows. The contraction of equation (28) (displayed in the next section) is an equation for q^2. If it is multiplied by Λ_{ij} and subtracted from equation (60), with $q^2\Lambda_{ij}$ written for Ω_{ij}, the result, after dividing through by q^2, is an equation for Λ_{ij}. It is considerably longer than (60) and not worth writing out. To determine Λ_{ij}, it is simpler to solve for Ω_{ij} and divide the result by q^2.

Thus, it is equation (60) that has been programmed for solution, along with equations (8), (9), and (28). Actually, for the modeling parameters that appear in both (28) and (60), the program version of (60) uses different names. In addition, the program version has two extra terms to allow direct comparison with earlier versions of the scale equation. Details are given in reference 1. In all the runs made so far, the renamed parameters have been given the same values as their prototypes, and the coefficients of the extra terms have been set to zero.

4.- THE BOUNDARY LAYER ASSUMPTIONS

In this initial attempt to study the behavior of the system consisting of equations (8), (9), (28), and (60), two-dimensional steady flow over a smooth flat plate is chosen. A Cartesian coordinate system, with ζ^1, ζ^2, ζ^3 denoted by x, y, z is used, taking the free stream parallel to the plate in the x direction and the y axis perpendicular to the plate so that the flow is envisioned in the x-y plane. Derivatives with respect to z, as well as to time t, are zero. The velocity components u_1, u_2, u_3 are denoted u, v, w. By the two-dimensional assumption, $\bar{w} = 0$, but this doesn't apply to w'.

Using x and y as subscripts to denote partial derivatives with respect to x and y, equation (8) becomes

$$\bar{u}_x + \bar{v}_y = 0 \tag{61}$$

Equation (9), for $i = 1$, becomes

$$\bar{u}\bar{u}_x + \bar{v}\bar{u}_y + (\overline{u'u'})_x + (\overline{u'v'})_y = -\bar{P}_x + \nu(\bar{u}_{xx} + \bar{u}_{yy}) \tag{62}$$

and for $i = 2$,

$$\bar{u}\bar{v}_x + \bar{v}\bar{v}_y + (\overline{u'v'})_x + (\overline{v'v'})_y = -\bar{P}_y + \nu(\bar{v}_{xx} + \bar{v}_{yy}) \tag{63}$$

Since the flow is two-dimensional, $i = 3$ in equation (9) is not relevant. At this point, the boundary layer assumptions are invoked.

At a high enough Reynolds number and in the absence of extremes in the pressure gradient, it is generally recognized that the influence of a flat plate on the flow along it is confined to a narrow boundary layer with the following characteristics. Derivatives with respect to y are large compared with derivatives with respect to x, $\bar{v}$ is small compared to $\bar{u}$, and the two effects balance each other so that the two terms in the continuity equation (61) are equal in magnitude.

In equation (62) then, $\bar{u}_{xx}$ is negligible compared to $\bar{u}_{yy}$ and, if it is assumed that the various components of $\overline{u_i'u_i'}$ are of the same order of magnitude, $(\overline{u'u'})_x$ is negligible compared to $(\overline{u'v'})_y$. On the other hand, the first two terms are comparable. Therefore, equation (62) can be approximated by

$$\bar{u}\bar{u}_x + \bar{v}\bar{u}_y = - \bar{P}_x - (\overline{u'v'})_y + \nu\bar{u}_{yy} \tag{64}$$

where $(\overline{u'v'})_y$ has been put on the other side of the equation since it is interpreted as representing a stress contributing to the rate of change of $\bar{u}$ along a streamline; that is, the left side.

The process of arriving at equation (64) from (62) can be formalized. Interpret the equations as being in terms of nondimensional variables so that ν stands for the reciprocal of the Reynolds number. Let δ be the nondimensional measure of a nominal boundary layer thickness, assumed small. Then $\bar{u}$ and $\bar{v}$ are taken to be of order δ^0 and δ^1, respectively; the operation of differentiation with respect to x and y is taken to be of order δ^0 and δ^{-1}, respectively; ν is taken to be of order δ^2. All this is standard in the theory of laminar boundary layers. The components of $\overline{u_i'u_j'}$ are taken to be of order δ^1. The terms that were dropped from equation (62) are, by these criteria, of order δ^1 or δ^2 and hence small compared to those found to be of order δ^0 ; that is, small compared to the rest of the terms except, perhaps, $\bar{P}_x$ which is considered in the next paragraph.

The terms of equation (63) are seen to be all of order δ^1 or smaller except $(\overline{v'v'})_y$ which is of order δ^0 and $\bar{P}_y$ which must also be of order δ^0. Thus,

$$(\overline{v'v'})_y = - \bar{P}_y$$

which can be integrated to give

$$\overline{v'v'} - (\overline{v'v'})_e = - \bar{P} + P_e$$

where the subscript stands for free stream conditions (i.e., the edge of the boundary layer). Solving for $\bar{P}$ and differentiating with respect to x gives

$$\bar{P}_x = P_{e_x} - (\overline{v'v'})_x + (\overline{v'v'})_{e_x}$$

for use in equation (64). However, the last two terms are of order δ^1 and so are dropped. From now on, $\bar{P}_x$ will be taken to mean P_{e_x}, the external pressure gradient.

The limitation on the pressure gradient needed to justify the boundary layer assumptions can now be made less vague by stating that $\bar{P}_x$ must be at most of order δ^0. It may, of course, be smaller, in which case it won't contribute appreciably to equation (64).

Similarly, it should not be assumed that the other terms in (64) are always of the order assigned to them. Indeed it is well known that in the main part of a turbulent boundary layer the term $\nu\bar{u}_{yy}$ is negligible compared $-(\overline{u'v'})_y$ but that the reverse is true in the sublayer next to the wall.

The same technique is applied to equation (28). The modeling constants are assumed to be of order δ^0. The definition of q^2 can be rewritten

$$q^2 = \overline{u'u'} + \overline{v'v'} + \overline{w'w'}$$

so q is of order $\delta^{\frac{1}{2}}$. To estimate the order of Λ, consider the contraction of equation (28):

$$(q^2)_t + \bar{u}^\ell (q^2)_{,\ell} + 2\overline{u'^\ell u'^n}\,\bar{u}_{n,\ell} - \text{VUU}\left[q\Lambda\left((q^2)^{,\ell} + 2(\overline{u'^\ell u'^n})_{,n}\right)\right]_{,\ell}$$

$$= -\text{PMU}\left[\Lambda^2\left(\bar{u}^\ell{}_{,n}(\overline{u'^m u'^n})_{,\ell} + \frac{1}{2}\bar{u}^m{}_{,n}(\overline{u'^\ell u'^n})_{,\ell}\right)\right]_{,m}$$

$$- 2\text{PUW}\left[q\Lambda(\overline{u'^\ell u'^n})_{,\ell}\right]_{,n} + \nu(q^2)^{,\ell}{}_{,\ell} - 2\frac{\nu a}{\Lambda^2}q^2 - 2b\frac{q}{\Lambda}q^2 \tag{65}$$

Among the unmodeled terms, the dominant one, after making the boundary layer assumptions, is a "production term," $2\overline{u'v'}\,\bar{u}_y$, of order δ^0. It is generally recognized that this term (which is ordinarily negative, since $\overline{u'v'}$ and $\bar{u}_y$ are usually of opposite sign) is principally balanced by "dissipation," that is, the last two terms in equation (65). If Λ is assigned the order δ^γ with γ to be determined, the first of the dissipation terms $2(\nu a/\Lambda^2)q^2$ has order $\delta^{3-2\gamma}$ and the second $2b(q/\Lambda)q^2$ has order $\delta^{3/2-\gamma}$. For either of these to balance the production term of order δ^0, $\gamma = 3/2$. The conclusion then is that taking Λ to be of order $\delta^{3/2}$ gives the proper ordering.

The order of all the terms in equation (65) can now be determined. Keeping only the largest gives

$$2\overline{u'v'}\,\bar{u}_y = -2\frac{\nu a}{\Lambda^2}q^2 - 2b\frac{q}{\Lambda}q^2 \tag{66}$$

This is a useful approximation for many purposes and is exploited

in the next section. (As is the case with $\overline{(u'v')}_y$ and $\nu\bar{u}_{yy}$ in equation (64), the two terms on the right of (66) are not important in the same part of the flow. Indeed, their ratio is

$$\frac{b}{a}\frac{q\Lambda}{\nu} = \frac{b}{a}\,Re_\Lambda$$

which varies from zero at the wall to values large compared to one.)

However, equation (66) is too crude an approximation for a study of the evolution of the boundary layer, so terms of order δ^1 are retained. Equation (65) becomes

$$\bar{u}(q^2)_x + \bar{v}(q^2)_y + 2\overline{u'u'}\,\bar{u}_x + 2\overline{u'v'}\,\bar{u}_y + 2\overline{v'v'}\,\bar{v}_y$$

$$- VUU\left[q\Lambda\left((q^2)_y + 2(\overline{v'v'})_y\right)\right]_y$$

$$= -\,2PUW\left[q\Lambda(\overline{v'v'})_y\right]_y + \nu(q^2)_{yy} - 2\,\frac{\nu a}{\Lambda^2}\,q^2 - 2b\,\frac{q}{\Lambda}\,q^2 \quad (67)$$

The changes in the modeling that have been introduced since reference 4 all drop out in the contraction of equation (28). Therefore, equation (67) is essentially the same as we would have written it then, with the exception of the terms $2\overline{u'u'}\,\bar{u}_x$ and $2\overline{v'v'}\,\bar{v}_y$ which we would have dropped, as do other workers in the field. The justification, if anything is said about it at all, it usually something like: "They are production terms; they are small compared to the other production term $(2\overline{u'v'}\,\bar{u}_y)$; therefore they are negligible." But if they are negligible, so are other terms in equation (67); the result is (66) again.

Note that insofar as these considerations apply to a comparison of the "small" production terms (which are also known as the stream-tube stretching terms) with the convection terms, $\bar{u}(q^2)_x$ and $\bar{v}(q^2)_y$, the reasoning is independent of the order of magnitude assigned to Λ, or for that matter to $\overline{u_i'u_j'}$. In fact, it is hard to conceive of a systematic ordering that would drop those production terms but retain the convection terms.

The boundary layer approximations can now be applied to equation (28) without contraction. It is found that $\overline{u'w'}$ and $\overline{v'w'}$ form an independent set and do not appear in the other equations; therefore they are ignored. So far, the equations have been written with the production terms and the triple correlation terms on the left with the convection terms, to emphasize their common origin in the term $(u^\ell u_i)_{,\ell}$ of equation (3). Henceforth, as is customary, only

the convection terms are written on the left, in parallel with equation (64).

In the boundary layer approximation, the terms multiplied by PUW appear in the same combinations as certain of the terms multiplied by VUU. Therefore it is convenient to introduce a new parameter:

$$\mathrm{VPW} = \mathrm{VUU} - \mathrm{PUW}$$

The final version of equation (28), then, consists of four equations:

$$\begin{aligned}\bar{u}(\overline{u'u'})_x + \bar{v}(\overline{u'u'})_y = &-2\overline{u'u'}\,\bar{u}_x - 2\overline{u'v'}\,\bar{u}_y + \mathrm{VUU}\left[q\Lambda(\overline{u'u'})_y\right]_y \\ &+ \frac{2}{3}\,\mathrm{GUA}(2\overline{u'u'}\,\bar{u}_x + 2\overline{u'v'}\,\bar{u}_y - \overline{v'v'}\,\bar{v}_y) + 2\,\mathrm{GUB}\,\overline{u'v'}\,\bar{u}_y \\ &+ 2\,\mathrm{GUC}\,q^2\bar{u}_x + \mathrm{GU2}\,\frac{q}{\Lambda}\left(\overline{u'u'} - \frac{q^2}{3}\right) + \nu(\overline{u'u'})_{yy} \\ &- 2\,\frac{\nu a}{\Lambda^2}\,\overline{u'u'} - 2b\,\frac{q}{\Lambda}\left(\beta\overline{u'u'} + \frac{1-\beta}{3}\,q^2\right)\end{aligned} \tag{68}$$

$$\begin{aligned}\bar{u}(\overline{v'v'})_x + \bar{v}(\overline{v'v'})_y = &-2\overline{v'v'}\,\bar{v}_y + \mathrm{VUU}\left[q\Lambda(\overline{v'v'})_y\right]_y + \\ &+ 2\,\mathrm{VPW}\left[q\Lambda(\overline{v'v'})_y\right]_y \\ &- \frac{2}{3}\,\mathrm{GUA}(\overline{u'u'}\,\bar{u}_x + \overline{u'v'}\,\bar{u}_y - 2\overline{v'v'}\,\bar{v}_y) - 2\,\mathrm{GUB}\,\overline{u'v'}\,\bar{u}_y \\ &+ 2\,\mathrm{GUC}\,q^2\,\bar{v}_y + \mathrm{GU2}\,\frac{q}{\Lambda}\left(\overline{v'v'} - \frac{q^2}{3}\right) + \nu(\overline{v'v'})_{yy} \\ &- 2\,\frac{\nu a}{\Lambda^2}\,\overline{v'v'} - 2b\,\frac{q}{\Lambda}\left(\beta\overline{v'v'} + \frac{1-\beta}{3}\,q^2\right)\end{aligned} \tag{69}$$

$$\bar{u}(\overline{w'w'})_x + \bar{v}(\overline{w'w'})_y = \mathrm{VUU}\left[q\Lambda(\overline{w'w'})_y\right]_y$$

$$- \frac{2}{3}\,\mathrm{GUA}(\overline{u'u'}\;\bar{u}_x + \overline{u'v'}\;\bar{u}_y + \overline{v'v'}\;\bar{v}_y) + \mathrm{GU2}\;\frac{q}{\Lambda}\left(\overline{w'w'} - \frac{q^2}{3}\right)$$

$$+ \nu(\overline{w'w'})_{yy} - 2\,\frac{\nu a}{\Lambda^2}\,\overline{w'w'} - 2b\,\frac{q}{\Lambda}\left(\beta\overline{w'w'} + \frac{1-\beta}{3}\,q^2\right) \quad (70)$$

$$\bar{u}(\overline{u'v'})_x + \bar{v}(\overline{u'v'})_y = -\,\overline{v'v'}\;\bar{u}_y + \mathrm{VUU}\left[q\Lambda(\overline{u'v'})_y\right]_y$$

$$+ \mathrm{VPW}\left[q\Lambda(\overline{u'v'})_y\right]_y - \frac{1}{4}\,\mathrm{PMU}\left[\Lambda^2\bar{u}_y(\overline{v'v'})_y\right]_y$$

$$+ \mathrm{GUA}\;\overline{v'v'}\;\bar{u}_y - \mathrm{GUB}(\overline{u'u'} - \overline{v'v'})\bar{u}_y$$

$$+ \mathrm{GUC}\;q^2\;\bar{u}_y + \mathrm{GU2}\;\frac{q}{\Lambda}\;\overline{u'v'} + \nu(\overline{u'v'})_{yy}$$

$$- 2\,\frac{\nu a}{\Lambda^2}\,\overline{u'v'} - 2b\,\frac{q}{\Lambda}\,\beta\,\overline{u'v'} \quad (71)$$

Applying the same approximations to equation (60) for Ω_{ij} involves no new concepts or problems. The smallest terms retained are again of the order of the convection terms and, hence, of Ω_{ij} itself, namely, $\delta^{5/2}$. As is the case for $\overline{u'w'}$ and $\overline{v'w'}$, Ω_{13} and Ω_{23} do not impinge on the other components and are ignored.

Some new notation is introduced, largely for convenience in programming:

$$\mathrm{EE} = \Omega_{11}$$

$$\mathrm{FF} = \Omega_{22}$$

$$\mathrm{GG} = \Omega_{33}$$

$$\mathrm{EF} = \Omega_{12}$$

where each letter pair is taken to be a single symbol; also

$$\mathrm{VOW} = \mathrm{VOM} - \mathrm{POW}$$

The final version of equation (60), then, consists of these equations:

$$\bar{u}EE_x + \bar{v}EE_y = -2EE\ \bar{u}_x - 2EF\ \bar{u}_y + VUU(q\Lambda EE_y)_y$$

$$+ POC\ \frac{q}{\Lambda}\ \Lambda_y{}^2\ EE$$

$$+ \frac{2}{3}\ GUA(2EE\ \bar{u}_x + 2EF\ \bar{u}_y - FF\ \bar{v}_y) + 2GUB\ EF\ \bar{u}_y$$

$$+ 2GUC\ \Omega\ \bar{u}_x + GU2\ \frac{q}{\Lambda}\left(EE - \frac{\Omega}{3}\right) + \nu EE_{yy}$$

$$- 2\ \frac{\nu AOM}{\Lambda^2}\ EE - 2BOM\ \frac{q}{\Lambda}\left(\beta\ EE + \frac{1 - \beta}{3}\ \Omega\right) \qquad (72)$$

$$\bar{u}FF_x + \bar{v}FF_y = -2FF\ \bar{v}_y + VUU(q\Lambda FF_y)_y + 2VPW(q\Lambda FF_y)_y$$

$$+ POC\ \frac{q}{\Lambda}\ \Lambda_y{}^2\ FF - \frac{2}{3}\ GUA(EE\ \bar{u}_x + EF\ \bar{u}_y - 2FF\ \bar{v}_y)$$

$$- 2GUB\ EF\ \bar{u}_y + 2GUC\ \Omega\ \bar{v}_y + GU2\ \frac{q}{\Lambda}\left(FF - \frac{\Omega}{3}\right)$$

$$+ \nu\ FF_{yy} - 2\ \frac{\nu AOM}{\Lambda^2}\ FF - 2BOM\ \frac{q}{\Lambda}\left(\beta\ FF + \frac{1 - \beta}{3}\ \Omega\right) \qquad (73)$$

$$\bar{u}GG_x + \bar{v}GG_y = VUU(q\Lambda GG_y)_y + POC\ \frac{q}{\Lambda}\ \Lambda_y{}^2\ GG$$

$$- \frac{2}{3}\ GUA(EE\ \bar{u}_x + EF\ \bar{u}_y + FF\ \bar{v}_y) + GU2\ \frac{q}{\Lambda}\left(GG - \frac{\Omega}{3}\right)$$

$$+ \nu\ GG_{yy} - 2\ \frac{\nu AOM}{\Lambda^2}\ GG - 2BOM\ \frac{q}{\Lambda}\left(\beta\ GG + \frac{1 - \beta}{3}\ \Omega\right) \qquad (74)$$

$$
\begin{aligned}
\bar{u}EF_x + \bar{v}EF_y = &- FF\ \bar{u}_y + VUU(q\Lambda FF_y)_y + VPW(q\Lambda FF_y)_y \\
&- \frac{1}{4}\ PMU(\Lambda^2 \bar{u}_y\ FF_y)_y + POC\ \frac{q}{\Lambda}\ \Lambda_y{}^2\ EF + GUA\ FF\ \bar{u}_y \\
&- GUB(EE - FF)\bar{u}_y + GUC\ \Omega\ \bar{u}_y + GU2\ \frac{q}{\Lambda}\ EF \\
&+ \nu\ EF_{yy} - 2\ \frac{\nu AOM}{\Lambda^2}\ EF - 2BOM\ \frac{q}{\Lambda}\ \beta\ EF
\end{aligned} \tag{75}
$$

5.- RESULTS

5.1 The Program

A computer program to solve the set consisting of equations (61), (64), and (68) through (75) by finite-difference methods is described in reference 1. A preliminary series of runs has been made to explore the nature of the scale tensor insofar as equation (60) expresses it. The description of the program in this paper is limited to those features actually used in the runs discussed.

All of the runs had a number of features in common. These include the following boundary conditions: all the dependent variables are zero at the wall, $y = 0$, and all go to zero in the free stream except $\bar{u}$ which is set to a constant, u_e, there and $\bar{v}$ which takes whatever values are needed to satisfy continuity. The pressure gradient $\bar{P}_x$ is set to zero.

The program computes two versions of Λ for use in the models at each point, Λ_r and Λ_e. (The subscripts are mnemonics for "ratio" and "empirical.") The first is defined by

$$\Lambda_r = \frac{\Omega}{q^2}$$

that is, Λ_r is the contraction of Λ_{ij}. The other is the function

$$\Lambda_e = \min \begin{cases} c\ \delta_{99} \\ d\ y \end{cases}$$

where c and d are constants and δ_{99} is the value of y for which $\bar{u} = .99\ u_e$. The values used for the constants are

$$c = .17$$
$$d = .65$$

which, in conjunction with the modeling parameters given below, are known to give good results for a flat plate boundary layer when Λ_e is used for Λ. Either Λ_r or Λ_e was used throughout any one run.

5.2 Evaluation of the Parameters

The modeling parameters appearing in the boundary layer equations for $\overline{u_i'u_j'}$, equations (68) through (71), were given the following values:

$$\begin{aligned} VUU &= .3 \\ VPW &= 0 \\ PMU &= 0 \\ GUA &= 0 \\ GUB &= 0 \\ GUC &= 0 \\ GU2 &= -1. \\ a &= 3.25 \\ b &= .125 \\ \beta &= 0 \end{aligned}$$

(Hence, the extra parameters of reference 1, referred to at the end of section 3, were given the values: VOM = .3 , VOW = POM = GOA = GOB = GOC = 0 , GO2 = -1. , SAM = TAL = 0 .) These parameters gave good results when used with Λ_e for Λ for the mean flow and for $\overline{u_i'u_j'}$ as they had in earlier programs. One exception to that statement is that the calculated maximum values of $\overline{u'u'}$, $\overline{v'v'}$, and $\overline{w'w'}$ are found to be approximately in the ratios

$$\overline{u'u'}:\overline{v'v'}:\overline{w'w'} = 2:1:1$$

which is not in agreement with experiment. This defect can easily be remedied by changing the values of GUA, GUB, and GUC, but that was not deemed necessary for this preliminary study, especially since various experiments do not agree on what the ratios should be.

There remains a group of three parameters, AOM, BOM, and POC, for which values are needed. Consider equation (67) in the free stream where $\bar{u}$ is constant and derivatives with respect to y are negligible:

$$\bar{u}(q^2)_x = -2 \frac{\nu a}{\Lambda^2} q^2 - 2b \frac{q}{\Lambda} q^2 \qquad (76)$$

The corresponding equation for Ω is obtained from the sum of equations (72), (73), and (74) (since $\Omega = EE + FF + GG$) with the same assumptions. It is

$$\bar{u}\,\Omega_x = -\frac{2\nu AOM}{\Lambda^2}\Omega - 2\ BOM \frac{q}{\Lambda}\Omega \qquad (77)$$

For this discussion, Λ is identified with Λ_r, so substituting $q^2\Lambda$ for Ω gives

$$\bar{u}\,\Lambda\,(q^2)_x + \bar{u}q^2\Lambda_x = -\frac{2\nu AOM}{\Lambda^2}\,q^2\Lambda - 2BOM\,\frac{q}{\Lambda}\,q^2\Lambda$$

Subtract equation (76) multiplied through by Λ from this and then divide the result through by q^2 :

$$\bar{u}\,\Lambda_x = -\frac{2\nu(AOM - a)}{\Lambda^2}\,\Lambda - 2(BOM - b)\,\frac{q}{\Lambda}\,\Lambda = -\frac{2\nu ATC}{\Lambda} - 2BTC\,q \qquad (78)$$

For high Reynolds numbers, $Re_\Lambda = q\Lambda/\nu$, the terms involving ν can be neglected. Then,

$$\bar{u}(q^2)_x = -2b\,\frac{q^3}{\Lambda} \qquad (79)$$

$$\bar{u}\,\Lambda_x = -2BTC\,q \qquad (80)$$

It is known (ref. 10) that in these circumstances q^2 decays like x^{-n} where n is probably around 1.25. For this to be compatible with equations (79) and (80), the relation

$$BTC = \left(\frac{1}{2} - \frac{1}{n}\right)b$$

must hold. Thus,

$$BOM = b + BTC = \left(\frac{3}{2} - \frac{1}{n}\right)b$$

For $n = 1.25$ and $b = .125$, these relations give $BTC = -.0375$ and $BOM = .0875$. As expected, Λ_x is positive so that Λ grows as q^2 decays.

While it is possible to approach the evaluation of AOM by considering equations (76) and (77) in the limit of low Reynolds numbers, a different flow situation is used here. It is easy to see that in the limit as y goes to zero, the dominant terms of equation (67) are $\nu(q^2)_{yy} - 2(\nu a/\Lambda^2)q^2$ so

$$(q^2)_{yy} = 2a\,\frac{q^2}{\Lambda^2} \qquad (81)$$

very near the wall. The corresponding equation for Ω is

$$\Omega_{yy} = 2AOM \frac{\Omega}{\Lambda^2} \tag{82}$$

With Λ (still identified with Λ_r) given by Ω/q^2, this pair of equations can be solved by assuming that q^2 varies like y^p and Ω varies like y^q. It is easily established that $q = p + 1$; that is, Λ is linear with y, a result in good agreement with views of the subject since Prandtl. If the slope of Λ as a function of y, for small y, is designated d (as it is for Λ_e), it is found further that $p(p-1) = (2a/d^2)$ or, taking the positive root,

$$p = \frac{1}{2}\left(1 + \sqrt{1 + \frac{8a}{d^2}}\right)$$

Also,

$$\frac{AOM}{a} = \frac{q(q-1)}{p(p-1)} = \frac{p+1}{p-1}$$

For $a = 3.25$ and $d = .65$, $p = 4.454$ and $AOM = 5.132$, approximately.

To obtain a value for POC, consider equations (68) through (71) keeping only the highest-order terms (as in eq. (76)) with the additional assumption that Re_Λ is large. The resulting set is equivalent, insofar as the modeling is the same, to the "superequilibrium" equations of reference 11. It is

$$\left(2 - \frac{4}{3}GUA - 2GUB\right)\overline{u'v'}\,\bar{u}_y =$$

$$= GU2\,\frac{q}{\Lambda}\left(\overline{u'u'} - \frac{q^2}{3}\right) - 2b\,\frac{q}{\Lambda}\left(\beta\,\overline{u'u'} + \frac{1-\beta}{3}q^2\right) \tag{83}$$

$$\left(\frac{2}{3}GUA + 2GUB\right)\overline{u'v'}\,\bar{u}_y =$$

$$= GU2\,\frac{q}{\Lambda}\left(\overline{v'v'} - \frac{q^2}{3}\right) - 2b\,\frac{q}{\Lambda}\left(\beta\,\overline{v'v'} + \frac{1-\beta}{3}q^2\right) \tag{84}$$

$$\frac{2}{3}\,\mathrm{GUA}\,\overline{u'v'}\,\bar{u}_y = \mathrm{GU2}\,\frac{q}{\Lambda}\left(\overline{w'w'} - \frac{q^2}{3}\right) - 2b\,\frac{q}{\Lambda}\left(\beta\,\overline{w'w'} + \frac{1-\beta}{3}\,q^2\right) \quad (85)$$

$$\left(1 - \mathrm{GUA} - \mathrm{GUB}\right)\overline{v'v'}\,\bar{u}_y =$$

$$= -\mathrm{GUB}\,\overline{u'u'}\,\bar{u}_y + \mathrm{GUC}\,q^2\bar{u}_y + \mathrm{GU2}\,\frac{q}{\Lambda}\,\overline{u'v'} - 2b\,\frac{q}{\Lambda}\,\beta\,\overline{u'v'} \quad (86)$$

These are regarded as a set determining the correlations as a function of Λ, $\bar{u}_y$, and the modeling constants. Although the set is nonlinear, due to the way q appears, the solution is readily obtained. Add equations (83), (84), and (85):

$$2\overline{u'v'}\,\bar{u}_y = -2b\,\frac{q}{\Lambda}\,q^2 \quad (87)$$

(Compare eq. (66).) Substituting this result back into equations (83), (84) and (85), they can easily be solved for $\overline{u'u'}$, $\overline{v'v'}$ and $\overline{w'w'}$ in terms of q^2. Incidentally, it is easy to see from equations (84) and (85) as they stand that it is GUB that controls whether or not $\overline{v'v'} = \overline{w'w'}$ in this approximation. If these values of the autocorrelations and the value of $\overline{u'v'}$ from equation (87) are substituted into (86), the result can be written

$$b(\mathrm{GU2} - 2b\beta)^2 q^2 = \Big\{ -\frac{1}{3}(1 - \mathrm{GUA})\left[\mathrm{GU2} + 2b(1-\beta)\right] + 2b\left[(1-\mathrm{GUA})\left(\frac{1}{3}\,\mathrm{GUA} + 2\,\mathrm{GUB}\right) - 2\,\mathrm{GUB}^2\right] + (\mathrm{GU2} - 2b\beta)\,\mathrm{GUC}\Big\}\,\Lambda^2\,\bar{u}_y^{\,2} \quad (88)$$

For the values of the parameters given above, this becomes simply

$$\frac{q^2}{\Lambda^2 u_y^{\,2}} = 2 \quad (89)$$

Substitution of the value of q thus determined back into the earlier equations allows the determination of the individual correlations.

It is known (refs. 8 and 11) that this approximation represents the situation in the law-of-the-wall region of a turbulent boundary layer. For example, equation (88) is in agreement with the known properties of that region in that the level of turbulence is roughly constant, Λ is proportional to y, and the velocity profile is logarithmic so that u_y is proportional to y^{-1}.

A similar analysis can be carried out on equations (72) through (75) but, in view of the variation of Λ in the law-of-the-wall region, terms involving Λ_y or $\Omega_{ij\,y}$ are retained, and it is assumed that each component of $\Omega_{ij\,y}$ is proportional to Λ_y. For the values of the parameters already mentioned, the equation corresponding to equation (89), to skip a few details, is

$$\frac{q^2}{\Lambda^2 u_y^{\,2}} = \frac{.55}{\left[.175 - (.3 + \mathrm{POC})\Lambda_y^{\,2}\right]\left[1 - (.3 + \mathrm{POC})\Lambda_y^{\,2}\right]^2}$$

or, using equation (89)

$$(.175 - z)(1 - z)^2 = .275$$

where $z = (.3 + \mathrm{POC})\Lambda_y^{\,2}$. The real root of this cubic is approximately $z = -.06669$. If Λ_y is assumed to have the value it has for $\Lambda = \Lambda_e$, namely, $\Lambda_y = .65$, it is found that $\mathrm{POC} = -.4578$.

To summarize, the following values have been found for the new parameters:

$$\mathrm{AOM} = 5.132$$

$$\mathrm{BOM} = .0875$$

$$\mathrm{POC} = -.4578$$

These, of course, are to be considered just first guesses with adjustments to be made after trial runs.

5.3 The Runs

A preliminary run was made to establish initial conditions for the other runs. The initial conditions for the preliminary run were those of a laminar boundary layer with a spot of low-level turbulence. The initial conditions for Ω_{ij} and the values of AOM, BOM, and POC were arbitrary. The run used $\Lambda = \Lambda_e$; it was stopped at a Reynolds number based on x, $Re_x = (u_e x/\nu)$, of 5 million.

After verifying that the profiles of $\bar{u}$, $\bar{v}$, and $\overline{u_i'u_j'}$ were representative of a fully developed turbulent boundary layer at that Reynolds number, they were used as initial conditions for the rest of the runs. Initial values of Ω_{ij} were set according to the formula

$$\Omega_{ij} = \Lambda_e \sigma_I \overline{u_i'u_j'} + \Lambda_e (1 - \sigma_I) \delta_{ij} \frac{q^2}{3}$$

where σ_I was either .9 or 1. For any value of σ_I, this formula guarantees that the Λ_r computed for these initial conditions is equal to Λ_e.

The first such run used the values of the new constants, AOM, BOM, and POC, found above, had $\sigma_I = 1$ and used $\Lambda = \Lambda_r$. As the solution proceeded downstream, Λ became quite small through most of the boundary layer. This emphasized the dissipation terms so that the level of turbulence, and with it the skin friction, dropped off very rapidly. A few trials with modified values of the new parameters showed similar behavior or the opposite, with the scale and turbulence levels growing much too large. It was quickly apparent that it would not be easy to find suitable values.

A series of runs was made with $\Lambda = \Lambda_e$ so that the turbulence level would remain normal while the effect of a variation of the new parameters on Ω_{ij} and on Λ_r could be monitored. A set of values that gave good results in this mode was

AOM = 5.65

BOM = .0875

POC = -.27

However, with $\Lambda = \Lambda_r$, the solution again showed rapidly increasing Λ and turbulence levels. Systematic changes in the parameters were tried to no avail. The results were unstable in the sense that a small change in one of the parameters could produce a large change in the solution. This indicates that even if one satisfactory run were obtained, it would no doubt be sensitive to initial conditions and other factors that should be extraneous to a useful calculation procedure.

These results are discouraging but not disastrous. For one thing, only the new parameters were varied. It is not necessarily true that values of the other parameters good for $\Lambda = \Lambda_e$ remain good for $\Lambda = \Lambda_r$. For another thing, the new modeling is deficient in a way that seems obvious in retrospect. A look at equation (53) suggests that the "POC" term could be expanded in the pattern of the "ATC" and "BTC" terms. That is, the "POC" term could be replaced by

$$\left\{\frac{2\nu APC}{\Lambda^2}\left(R_{ij} - N_{ij}\right) + 2BPC\,\frac{q}{\Lambda}\left[\beta\left(R_{ij} - N_{ij}\right) + \frac{1-\beta}{3}\,g_{ij}\left(R^{\ell}{}_{\ell} - N^{\ell}{}_{\ell}\right)\right]\right\} \frac{\partial\Lambda}{\partial\zeta^{\ell}}\,\frac{\partial\Lambda}{\partial\zeta^{\ell}}$$

It is expected that the presence of β would have a minor effect, but the first term might turn out to be more important.

In the discussion above, the possibility of using the low Reynolds number limit of equations (76) and (78) for the decay of turbulence in the free stream to determine ATC (and hence AOM) was glossed over. It turns out that if that is done, and it is noted that q^2 should decay like $x^{-5/2}$ in that limit (ref. 9), the value $AOM/a = .8$ is found. This value would not work near the wall since it leads to a singularity at $y = 0$. But when the proposed new term is included in the model, then AOM will be replaced by $AOM - APC\,\Lambda_y^2$ in the calculation near the wall, but the calculation for the free stream will remain unchanged. Thus the conflict in the two ways of determining AOM will be removed.

From the numerical results obtained so far, it appears that this change will definitely improve the outcome. Whether it will be sufficient to overcome the difficulties remains to be seen. Unfortunately, this seemingly obvious improvement was not thought of until this paper was in preparation, too late to include it in the program under this contract.

5.4 The Nature of the Tensor

As stated in the introduction, a main purpose in developing the program was to elucidate the structure of turbulence, at least as modeled, through an examination of the scale tensor. One particular aspect was studied in these preliminary runs. It has been proposed (ref. 7) that the "quasi-isotropic" assumption is a useful approximation. That assumption is

$$\Omega_{ij} = \frac{1}{3}\,g_{ij}\Omega + S_L\left(\overline{u_i' u_j'} - \frac{1}{3}\,g_{ij}\,q^2\right) \tag{90}$$

where S_L, a length, is a function of position but independent of i,j. If equation (90) is valid, then instead of solving for all the relevant components of Ω_{ij} en route to determining Λ_r, it is sufficient to solve for the two quantities, Ω and S_L.

Equation (90) was tested by forming the ratio

$$S_{L_{ij}} = \frac{\Omega_{ij} - \frac{1}{3} g_{ij} \Omega}{u_i' u_j' - \frac{1}{3} g_{ij} q^2} \quad \text{(not summed)}$$

at each point for each pair i,j . If the various $S_{L_{ij}}$ at a given point are equal, the hypothesis is verified. It was found that the diagonal elements $(i = j)$ were very close to each other, generally within one part in a thousand, but the off-diagonal value tended to be about ten percent higher.

A further hypothesis is that

$$S_L = \sigma \frac{\Omega}{q^2} = \sigma \Lambda_r$$

where σ is a constant. This too was tested, using the value found on the diagonal of $S_{L_{ij}}$ for S_L . It was found that σ was almost always in the range from .85 to .95, and most often between .88 and .92. Except for some transients dependent on the initial conditions, these results concerning S_L and σ were remarkably constant across the runs, independent of whether $\Lambda = \Lambda_r$ or $\Lambda = \Lambda_e$ was used and independent of the choice of parameters. Too much shouldn't be made of these observations until the hypotheses are tested in other flow situations. If they hold up, however, it appears that if the accuracy of the first is acceptable, then setting $\sigma = .9$ is equally valid and only one differential equation need be solved to determine Λ_r .

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DEC 11 1979